THE AUTOMORPHISM GROUP OF A METROPOLIS-ROTA IMPLICATION ALGEBRA

COLIN G. BAILEY AND JOSEPH S. OLIVEIRA

Abstract. We discuss the group of automorphisms of a general MR-algebra. We develop several functors between implication algebras and cubic algebras. These allow us to generalize the notion of inner automorphism. We then show that this group is always isomorphic to the group of inner automorphisms of a filter algebra.

1. Introduction

Cubic implication algebras are an algebraic generalization of the algebra of faces of an n-cube as introduced by Metropolis & Rota in [4]. From their paper and further work of the authors ([1, 2]) much is known about the structure of cubic implication algebras. A survey of this material may be found in [3].

The group of automorphisms of a face poset of an n-cube is well-known to be $\mathbb{Z}_2^n \rtimes S_n$. We will show that in every cubic implication algebra there is a subgroup of definable automorphisms, known as *inner automorphisms*, that corresponds to the $\mathbb{Z}_2^n$ portion of this group.

Associated with any cubic implication algebra is a minimal enveloping Metropolis-Rota implication algebra (usually abbreviated as MR-algebra). Notions such as congruences and automorphisms lift from the cubic implication algebra to its envelope. Thus a major part of characterizing the automorphism group of a cubic implication algebra is the special case of MR-algebras. Earlier work ([2]) has dealt with proper subclasses – interval algebras and filter algebras. Herein we are interested in automorphism groups of arbitrary MR-algebras. By careful construction of subalgebras we are able to lift results from filter and interval algebras to the general case.

We begin with some definitions and basic results from [1].

Definition 1.1. *A* cubic implication algebra *is a join semi-lattice with one and a binary operation* Δ *satisfying the following axioms:*

a. if $x \leq y$ *then* $\Delta(y, x) \vee x = y$;
b. if $x \leq y \leq z$ *then* $\Delta(z, \Delta(y, x)) = \Delta(\Delta(z, y), \Delta(z, x))$;
c. if $x \leq y$ *then* $\Delta(y, \Delta(y, x)) = x$;
d. if $x \leq y \leq z$ *then* $\Delta(z, x) \leq \Delta(z, y)$;

Let $x \to y = \Delta(\mathbf{1}, \Delta(x \vee y, y)) \vee y$ *for any* x, y *in* $\mathcal{L}$. *Then:*

e. $(x \to y) \to y = x \vee y$;
f. $x \to (y \to z) = y \to (x \to z)$;

Date: 2009, February 6.
1991 *Mathematics Subject Classification.* 08A35, 06A12.
Key words and phrases. cubes, automorphisms, implication algebras.

Definition 1.2. *An* MR-algebra *is a cubic implication algebra satisfying the MR-axiom:*
if $a, b < x$ then

$$\Delta(x, a) \vee b < x \text{ iff } a \wedge b \text{ does not exist.}$$

Definition 1.3. *Let $\mathcal{L}$ be a cubic implication algebra. Then for any $x, y \in \mathcal{L}$ we define the (partial) operations ˆ (*caret*) and $*$ by:*

(a)

$$x \,\hat{}\, y = x \wedge \Delta(x \vee y, y)$$

whenever this meet exists.

(b)

$$x * y = x \vee \Delta(x \vee y, y)$$

Lemma 1.4. *Every cubic implication algebra is an implication algebra.*

Proof. It suffices to note that $x \to x = \Delta(\mathbf{1}, \Delta(x, x)) \vee x = \Delta(\mathbf{1}, x) \vee x = \mathbf{1}$. □

Lemma 1.5. *If $\mathcal{L}$ is a cubic implication algebra then $\mathcal{L}$ is an MR-algebra iff the caret operation is total.*

Proof. If $\mathcal{L}$ is an MR-algebra, then for any a, b we have $a \vee b = a \vee b$ and so $a \wedge \Delta(a \vee b, b) = a \,\hat{}\, b$ exists.

Conversely, suppose caret is total. If $a \wedge b$ exists, then $x \geq a \vee \Delta(x, b) \geq (a \wedge b) \vee \Delta(x, a \wedge b) = x$.

Now suppose that $a \vee \Delta(x, b) = x$. There are two cases

- if $a \vee b$ is one of a or b, then a and b are comparable and the meet clearly exists.
- Otherwise $a, b < a \vee b$. By [1] theorem 4.3 we must have

$$a \vee \Delta(a \vee b, b) = a \vee b. \tag{1}$$

Then we have

$$\begin{aligned} a \,\hat{}\, \Delta(a \vee b, b) &= a \wedge \Delta(a \vee \Delta(a \vee b, b), \Delta(a \vee b, b)) && \text{by definition} \\ &= a \wedge \Delta(a \vee b, \Delta(a \vee b, b)) && \text{by (1)} \\ &= a \wedge b. \end{aligned}$$

□

Definition 1.6. *Let $\mathcal{L}$ be a cubic implication algebra and $a, b \in \mathcal{L}$. Then*

$$a \preccurlyeq b \text{ iff } \Delta(a \vee b, a) \leq b$$
$$a \sim b \text{ iff } \Delta(a \vee b, a) = b.$$

Lemma 1.7. *Let $\mathcal{L}$, a, b be as in the definition above. Then*

$$a \preccurlyeq b \text{ iff } b = (b \vee a) \wedge (b \vee \Delta(\mathbf{1}, a)).$$

Proof. See [1] lemmas 2.7 and 2.12. □

Proposition 1.8. *Let $\mathcal{L}$ be a cubic implication algebra, and p, q in $\mathcal{L}$ are such that $p \preccurlyeq q$ and $p \wedge q$ exists. Then $p \leq q$.*

Proof. We have $\Delta(p \vee q, p) \leq q$ as $p \preccurlyeq q$. Let $a = p \wedge q$. Then

$$\begin{aligned} a &\leq q \\ \Delta(p \vee q, a) &\leq \Delta(p \vee q, p) \leq q \\ \text{Hence} \qquad p \vee q &= a \vee \Delta(p \vee q, a) \\ &\leq q \end{aligned}$$

and so $p \leq q$. □

Corollary 1.9. *Let $\mathcal{L}$ be a cubic implication algebra, and p, q in $\mathcal{L}$ are such that $p \simeq q$ and $p \wedge q$ exists. Then $p = q$.*

Also from [1] (lemma 2.7c for transitivity) we know that $\sim$ is an equivalence relation on all cubic implication algebras, and is a congruence on the structure $\langle \mathcal{L}, \hat{}\,, *, \mathbf{1} \rangle$. The quotient is naturally an implication algebra.

As part of the representation theory in [1] we had the following definitions and lemma:

Definition 1.10. *Let $\mathcal{L}$ be a cubic implication algebra. Then*

(a) $\mathcal{L}_a = \{\Delta(y, x) \mid a \leq x \leq y\}$ *for any* $a \in \mathcal{L}$. $\mathcal{L}_a$ *is the* localization *of* $\mathcal{L}$ *at* a.
(b) $k_a(y) = (\Delta(\mathbf{1}, y) \vee a)a$ *for any* $y \in \mathcal{L}_a$.
(c) $\ell_a(y) = y \vee a$ *for any* $y \in \mathcal{L}_a$.

Lemma 1.11. *Let $\mathcal{L}$ be any cubic implication algebra. Then*

(a) $\mathcal{L}_a$ *is an atomic MR-algebra, and hence isomorphic to an interval algebra.*
(b) $k_a(x) \leq \ell_a(x)$ *for any* $x \in \mathcal{L}_a$.
(c) If $x, y \in \mathcal{L}_a$ *then*

$$\begin{aligned} x = y &\iff k_a(x) = k_a(y) \text{ and } \ell_a(x) = \ell_a(y) \\ &\iff x \vee a = y \vee a \text{ and } x \vee \Delta(\mathbf{1}, a) = y \vee \Delta(\mathbf{1}, a). \end{aligned}$$

(d) If $p \geq q \geq a$ *then there exists a unique* $z \in \mathcal{L}_a$ *such that* $\ell_a(z) = p$ *and* $k_a(z) = q$.
(e) $x \in \mathcal{L}_a$ *iff* $a \preccurlyeq x$.

Proof. See [1]. □

2. Construction of cubic implication algebras

There is a general construction of cubic implication algebras from implication algebras.

Let $\mathcal{I}$ be an implication algebra. We define

$$\mathscr{I}(\mathcal{I}) = \{\, \langle a, b \rangle \mid a, b \in \mathcal{I}, a \vee b = 1 \text{ and } a \wedge b \text{ exists}\}$$

ordered by

$$\langle a, b \rangle \leq \langle c, d \rangle \ \text{ iff } a \leq c \text{ and } b \leq d.$$

This is a partial order that is an upper semi-lattice with join defined by

$$\langle a, b \rangle \vee \langle c, d \rangle = \langle a \vee c, b \vee d \rangle$$

and a maximum element $\mathbf{1} = \langle 1, 1 \rangle$.

We can also define a Δ function by

$$\text{if } \langle c, d \rangle \leq \langle a, b \rangle \text{ then } \Delta(\langle a, b \rangle, \langle c, d \rangle) = \langle a \wedge (b \to d), b \wedge (a \to c) \rangle\,.$$

We note the natural embedding of $\mathcal{I}$ into $\mathscr{I}(\mathcal{I})$ given by

$$e_{\mathcal{I}}(a) = \langle 1, a\rangle\,.$$

Note that in an implication algebra $a \vee b = \mathbf{1}$ iff $a \to b = b$ iff $b \to a = a$.

Also $\Delta(\mathbf{1}, \bullet)$ is very simply defined as it is exactly $\langle a, b\rangle \mapsto \langle b, a\rangle$.

It is not hard to show that $\mathscr{I}(\mathcal{I})$ is a cubic implication algebra, and is an MR-algebra iff $\mathcal{I}$ is a lattice.

$\mathcal{I}$ is a lattice produces two cases – either there is a least element or not.

In the first of these cases $\mathcal{I}$ is a Boolean algebra and $\mathscr{I}(B)$ is naturally isomorphic to the algebra of closed intervals of B – we call these algebras *interval algebras*.

In the second case $\mathcal{I}$ is isomorphic to an ultrafilter of some Boolean algebra B and we get a *filter algebra* which can be embedded as an upwards closed MR-subalgebra of $\mathscr{I}(B)$.

Every interval algebra is also a filter algebra, as any Boolean algebra B is an ultrafilter in $B \times \mathbf{2}$.

MR-algebras that are isomorphic to filter algebras have an automorphism group that splits as a twisted product of a group of inner automorphisms with the group of automorphisms of the filter. The former group is isomorphic to a Boolean algebra that is naturally definable from the filter. It is important in what follows to know that there are many MR-algebras that are filter algebras. In particular ones that are countably presented.

Definition 2.1. *Let $\mathcal{L}$ be a cubic implication algebra.*

(a) Let $A \subseteq \mathcal{L}$. A is a presentation *of $\mathcal{L}$ iff*

$$\mathcal{L} = \bigcup_{a\in A} \mathcal{L}_a.$$

(b) $\mathcal{L}$ is countably presented *iff there is a countable set presenting $\mathcal{L}$.*

Theorem 2.2. *Let $\mathcal{M}$ be a countably presented MR-algebra. Then $\mathcal{M}$ is a filter algebra.*

Proof. Let $A = \{a_i \mid i \in \omega\}$ be a countable set that presents $\mathcal{M}$. Define the sequence $\langle b_n \mid n \in \omega\rangle$ by

$$b_0 = a_0$$
$$b_{n+1} = b_n \mathbin{\hat{}} a_{n+1}.$$

Then we have $b_{n+1} \leq b_n$ and $b_{n+1} \preccurlyeq a_{n+1}$ so that

$$\mathcal{M} = \bigcup_{n=0}^{\infty} \mathcal{M}_{a_n} = \bigcup_{n=0}^{\infty} \mathcal{M}_{b_n}$$

and $\{x \mid \exists n\ x \geq b_n\}$ is therefore a generating filter for $\mathcal{M}$. □

3. Some Facts about Filter Algebras

Let $\mathscr{I}(\mathscr{F})$ be a filter algebra. We recall some earlier results about automorphisms of $\mathscr{I}(\mathscr{F})$ from [2].

Definition 3.1 ([2, Definitions 22, 34])**.** *Let $\mathcal{L} = \mathscr{I}(\mathscr{F})$ be any filter algebra. Let $\mathscr{G} \subseteq \mathcal{L}$ be a filter.*

(a) $[\![\mathscr{G}]\!]$ is the subalgebra generated by $\mathscr{G}$;

(b) $\widehat{\mathscr{G}} = \{\Delta(x, y) \mid x, y \in \mathscr{G} \text{ and } y \leq x\}$*;*
(c) $\mathscr{G}$ *is a* generating filter *or* g-filter *iff* $[\![\mathscr{G}]\!] = \mathcal{L}$.

Lemma 3.2 ([2, Theorem 29])**.** *Let* $\mathscr{G}$ *be a filter in a cubic implication algebra* $\mathcal{L}$*. Then*

$$[\![\mathscr{G}]\!] = \widehat{\mathscr{G}}.$$

Definition 3.3 ([2, Definition 35])**.** *Let* $\mathcal{M}$ *be an MR-algebra, and* $\mathscr{F}$ *be a generating filter in* $\mathcal{M}$*. Then for all* $x \in \mathcal{M}$ *we let* $\alpha_{\mathscr{F}}(x)$ *and* $\beta_{\mathscr{F}}(x)$ *be the unique elements in* $\mathscr{F}$ *such that* $x = \Delta(\alpha_{\mathscr{F}}(x), \beta_{\mathscr{F}}(x))$*.*

Theorem 3.4 ([2, Corollary 43])**.** *Let* $\mathcal{M}$ *be an MR-algebra, and* $\mathscr{F}$*,* $\mathscr{G}$ *be two generating filters in* $\mathcal{M}$*. Then the function* $x \mapsto \beta_{\mathscr{G}}(x)$ *from* $\mathscr{F}$ *to* $\mathscr{G}$ *is a one-one onto implication homomorphism.*

Definition 3.5 ([2, Definition 47])**.** *Let* $\mathcal{M}$ *be an MR-algebra, and* $\mathscr{F}$*,* $\mathscr{G}$ *be two generating filters in* $\mathcal{M}$*.*

A function $f\colon \mathcal{M} \to \mathcal{M}$ *is a* filter automorphism based on $\langle \mathscr{F}, \mathscr{G}\rangle$ *iff*
(a) f *is an automorphism of* $\mathcal{M}$*;*
(b) $f[\mathscr{F}] = \mathscr{G}$*;*
(c) for all $x \in \mathscr{F}$ $x \sim f(x)$*.*
Filter automorphisms are also called inner automorphisms *and the set of all inner automorphisms is denoted by* $\mathrm{Inn}(\mathcal{M})$*.*

Lemma 3.6 ([2, Lemma 48, Definition 49])**.** *Let* $\mathcal{M}$ *be an MR-algebra and* $\mathscr{F}$*,* $\mathscr{G}$ *be two generating filters in* $\mathcal{M}$*. Then there is a unique filter automorphism* f *such that* $f[\mathscr{F}] = \mathscr{G}$*. This automorphism is denoted by* $\varphi_{\langle \mathscr{F}, \mathscr{G}\rangle}$*.*

Lemma 3.7 ([2, Lemma 51])**.** *Let* $\mathcal{M}$ *be a filter algebra. Let* $\mathscr{F}$ *be a generating filter in* $\mathcal{M}$*. Let* f *be a cubic automorphism such that* $x \sim f(x)$ *for all* $x \in \mathcal{M}$*. Then*
(a) $f[\mathscr{F}]$ *is a generating filter; and*
(b) $f = \varphi_{\langle \mathscr{F}, f[\mathscr{F}]\rangle}$*.*

From this lemma it is easy to show that the set of filter automorphisms is a group, but we also want to know that it has 2-torsion.

Lemma 3.8 ([2, Lemma 58, Corollary 59])**.** *Let* $\mathcal{M}$ *be an MR-algebra and* $\mathscr{F}, \mathscr{G}$ *be two generating filters in* $\mathcal{M}$*. Then*

$$\varphi_{\langle \mathscr{F}, \mathscr{G}\rangle} = \varphi_{\langle \mathscr{G}, \mathscr{F}\rangle}$$

and hence

$$\varphi^{-1}_{\langle \mathscr{F}, \mathscr{G}\rangle} = \varphi_{\langle \mathscr{F}, \mathscr{G}\rangle}.$$

Extending the ideas of [2] we define operations and properties of filters and g-filters, and can prove certain consequences. These results will appear in more detail in a later paper.

Definition 3.9. *Let* $\mathscr{G} \subseteq \mathscr{F}$ *be two* $\mathcal{L}$*-filters. Then*
(a) $\mathscr{G} \supset \mathscr{F} = \bigcap \{\mathscr{H} \mid \mathscr{H} \vee \mathscr{G} = \mathscr{F}\}$*;*
(b) $\mathscr{G} \Rightarrow \mathscr{F} = \bigvee \{\mathscr{H} \mid \mathscr{H} \subseteq \mathscr{F} \text{ and } \mathscr{H} \cap \mathscr{G} = \{\mathbf{1}\}\}$*;*
(c) $\mathscr{G} \to \mathscr{F} = \{h \in \mathscr{F} \mid \forall g \in \mathscr{G}\ h \vee g = \mathbf{1}\}$*.*

Lemma 3.10.

(a) $\mathscr{G} \to \mathscr{F} = \mathscr{G} \Rightarrow \mathscr{F}$.
(b) $\mathscr{G} \supset \mathscr{F} = \mathscr{G} \to \mathscr{F}$.

Particular amongst all filters are Boolean filters.

Definition 3.11. *Let $\mathscr{F}$ be a g-filter. Then*
(a) $\mathscr{G}$ *is* weakly $\mathscr{F}$-Boolean *iff* $\mathscr{G} \subseteq \mathscr{F}$ *and* $(\mathscr{G} \to \mathscr{F}) \to \mathscr{F} = \mathscr{G}$.
(b) $\mathscr{G}$ *is* weakly Boolean *iff there is some g-filter containing* $\mathscr{G}$ *and* $\mathscr{G}$ *is* $\mathscr{H}$*-Boolean for all such g-filters* $\mathscr{H}$.
(c) $\mathscr{G}$ *is* $\mathscr{F}$-Boolean *iff* $\mathscr{G} \subseteq \mathscr{F}$ *and* $\mathscr{G} \vee (\mathscr{G} \to \mathscr{F}) = \mathscr{F}$.
(d) $\mathscr{G}$ *is* Boolean *iff there is some g-filter containing* $\mathscr{G}$ *and* $\mathscr{G}$ *is* $\mathscr{H}$*-Boolean for all such g-filters* $\mathscr{H}$.

Theorem 3.12. *Let $\mathscr{G}$ be $\mathscr{F}$-Boolean for some g-filter $\mathscr{F}$. Then $\mathscr{G}$ is Boolean.*

Now we can define a Δ operation on filters and this allows recovery of a g-filter some certain fragments.

Definition 3.13. *Let $\mathscr{G} \subseteq \mathscr{F}$. Then*

$$\Delta(\mathscr{G}, \mathscr{F}) = \Delta(\mathbf{1}, \mathscr{G} \to \mathscr{F}) \vee \mathscr{G}.$$

Theorem 3.14. *If $\mathscr{H}$, $\mathscr{F}$ are g-filters then $\mathscr{G} = \mathscr{F} \cap \mathscr{H}$ is $\mathscr{F}$-Boolean and $\mathscr{H} = \Delta(\mathscr{G}, \mathscr{F})$.*

Conversely, if $\mathscr{G}$ is $\mathscr{F}$-Boolean then $\mathscr{H} = \Delta(\mathscr{G}, \mathscr{F})$ is a g-filter and $\mathscr{G} = \mathscr{F} \cap \mathscr{H}$.

4. Some Category Theory

The operation $\mathscr{I}$ is a functor where we define $\mathscr{I}(f) \colon \mathscr{I}(\mathcal{I}_1) \to \mathscr{I}(\mathcal{I}_2)$ by

$$\mathscr{I}(f)(\langle a, b\rangle) = \langle f(a), f(b)\rangle$$

whenever $f \colon \mathcal{I}_1 \to \mathcal{I}_2$ is an implication morphism.

The relation $\sim$ defined above gives rise to a functor $\mathscr{C}$ on cubic implication algebras. In order to show that it is well-defined this we need the following lemma.

Lemma 4.1. *Let $\phi \colon \mathcal{L}_1 \to \mathcal{L}_2$ be a cubic homomorphism. Let $a, b \in \mathcal{L}_1$. Then*

$$a \sim b \Rightarrow \phi(a) \sim \phi(b).$$

Proof.

$$\begin{aligned} a \sim b &\iff \Delta(a \vee b, a) = b \\ &\Rightarrow \phi(\Delta(a \vee b, a)) = \phi(b) \\ &\iff \Delta(\phi(a) \vee \phi(b), \phi(a)) = \phi(b) \\ &\iff \phi(a) \sim \phi(b). \end{aligned}$$

□

$\mathscr{C}$ is defined by

$$\mathscr{C}(\mathcal{L}) = \mathcal{L}/\sim$$
$$\mathscr{C}(\phi)([x]) = [\phi(x)].$$

There are several natural transformations here. The basic ones are $e \colon \mathrm{ID} \to \mathscr{I}$ and $\eta \colon \mathrm{ID} \to \mathscr{C}$, defined by

$$e_{\mathcal{I}}(x) = \langle \mathbf{1}, x\rangle$$
$$\eta_{\mathcal{L}}(x) = [x].$$

The following diagram commutes:

$$\begin{tikzcd} \mathcal{I}_1 \arrow[r, "\phi"] \arrow[d, "e_{\mathcal{I}_1}"'] & \mathcal{I}_2 \arrow[d, "e_{\mathcal{I}_2}"'] \\ \mathscr{I}(\mathcal{I}_1) \arrow[r, "\mathscr{I}(\phi)"'] & \mathscr{I}(\mathcal{I}_2) \end{tikzcd}$$

as for $x \in \mathcal{I}_1$, we have

$$\begin{aligned} e_{\mathcal{I}_2}(\phi(x)) &= \langle \mathbf{1}, \phi(x) \rangle \\ &= \langle \phi(\mathbf{1}), \phi(x) \rangle \\ &= \mathscr{I}(\phi)(\langle \mathbf{1}, x \rangle) \\ &= \mathscr{I}(\phi) e_{\mathcal{I}_1}(x). \end{aligned}$$

The following diagram commutes:

$$\begin{tikzcd} \mathcal{L}_1 \arrow[r, "\phi"] \arrow[d, "\eta_{\mathcal{L}_1}"'] & \mathcal{L}_2 \arrow[d, "\eta_{\mathcal{L}_2}"'] \\ \mathscr{C}(\mathcal{L}_1) \arrow[r, "\mathscr{C}(\phi)"'] & \mathscr{C}(\mathcal{L}_2) \end{tikzcd}$$

as for $x \in \mathcal{L}_1$, we have

$$\begin{aligned} \eta_{\mathcal{L}_2}(\phi(x)) &= [\phi(x)] \\ &= \mathscr{C}(\phi)([x]) \\ &= \mathscr{C}(\phi) \eta_{\mathcal{L}_1}(x). \end{aligned}$$

Then we get the composite transformation $\iota \colon \mathrm{ID} \to \mathscr{C}\mathscr{I}$ defined by

$$\iota_{\mathcal{I}} = \eta_{\mathscr{I}(\mathcal{I})} \circ e_{\mathcal{I}}.$$

By standard theory this is a natural transformation. It is easy to see that $e_{\mathcal{I}}$ is an embedding, and that $\eta_{\mathcal{L}}$ is onto.

Theorem 4.2. *$\iota_{\mathcal{I}}$ is an isomorphism.*

Proof. Let $x, y \in \mathcal{I}$ and suppose that $\iota(x) = \iota(y)$. Then

$$\begin{aligned} \iota(x) &= \eta_{\mathscr{I}(\mathcal{I})}(e_{\mathcal{I}}(x)) \\ &= [\langle \mathbf{1}, x \rangle] \\ &= [\langle \mathbf{1}, y \rangle]. \end{aligned}$$

Thus $\langle \mathbf{1}, x \rangle \sim \langle \mathbf{1}, y \rangle$. Now

$$\begin{aligned} \Delta(\langle \mathbf{1}, x \rangle \vee \langle \mathbf{1}, y \rangle, \langle \mathbf{1}, y \rangle) &= \Delta(\langle \mathbf{1}, x \vee y \rangle, \langle \mathbf{1}, y \rangle) \\ &= \langle (x \vee y) \to y, x \vee y \rangle. \end{aligned}$$

This equals $\langle \mathbf{1}, x \rangle$ iff $x = x \vee y$ (so that $y \le x$) and $(x \vee y) \to y = \mathbf{1}$ so that $y = x \vee y$ and $x \le y$. Thus $x = y$. Hence ι is one-one.

It is also onto, as if $z \in \mathscr{C}\mathscr{I}(\mathcal{I})$ then we have $z = [w]$ for some $w \in \mathscr{I}(\mathcal{I})$. But we know that $w = \langle x, y\rangle \sim \langle \mathbf{1}, x \wedge y\rangle$ – since $\Delta(\langle \mathbf{1}, y\rangle, \langle \mathbf{1}, x \wedge y\rangle) = \langle x, y\rangle$ – and so $z = [\langle \mathbf{1}, x \wedge y\rangle] = \eta_{\mathscr{I}(\mathcal{I})}(e_{\mathcal{I}}(x \wedge y))$. □

We note that there is also a natural transformation $\kappa\colon \mathrm{ID} \to \mathscr{I}\mathscr{C}$ defined by

$$\kappa_{\mathcal{L}} = e_{\mathscr{C}(\mathcal{L})} \circ \eta_{\mathcal{L}}.$$

In general this is not an isomorphism as there are MR-algebras $\mathcal{M}$ that are not filter algebras, but $\mathscr{I}(\mathscr{C}(\mathcal{M}))$ is always a filter algebra.

We also note that $\iota_{\mathscr{C}(\mathcal{L})} = \mathscr{C}(\kappa_{\mathcal{L}})$ for all cubic implication algebras $\mathcal{L}$. The pair $\mathscr{I}$ and $\mathscr{C}$ do not form an adjoint pair.

5. Automorphisms

In [2] we showed that the automorphism group of a filter algebra $\mathscr{I}(\mathscr{F})$ decomposes into a group of *inner* automorphisms and the group of implication automorphisms of $\mathscr{F}$. In the special case of an interval algebra $\mathscr{I}(B)$ the former group is isomorphic to $\langle B, 0, +\rangle$. For arbirary $\mathscr{I}(\mathscr{F})$ the group of inner automorphisms is isomorphic to a 2-torsion group of subfilters of $\mathscr{F}$.

Now we want to look at the most general case of MR-algebras and determine some of the structure of the automorphism group.

Let $\mathcal{M}$ be any MR-algebra. The functor $\mathscr{C}$ induces a group homomorphism $\mathscr{C}\colon \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathscr{C}(\mathcal{M}))$. We want to look at the kernel of $\mathscr{C}$.

The method is somewhat indirect and first we consider what $\mathscr{C}$ does in the case that $\mathcal{M}$ is a filter algebra.

5.1. $\mathscr{C}$ on Filter algebras.

The kernel of $\mathscr{C}$ on a filter algebra is relatively easy to compute.

Theorem 5.1. *Let $\mathcal{M}$ be a filter algebra. Then*

$$\ker(\mathscr{C}) = \mathrm{Inn}(\mathcal{M}).$$

Proof. Let ϕ be any inner automorphism. Then we know that $x \sim \phi(x)$ for all x so that $[x] = [\phi(x)] = \mathscr{C}(\phi)([x])$ for all x. Thus $\phi \in \ker(\mathscr{C})$.

Conversely if $\phi \in \ker(\mathscr{C})$ then $x \sim \phi(x)$ for all x. Then by lemma 3.7 we know that ϕ is an inner automorphism. □

We recall that if $\mathcal{M}$ is a filter algebra and $\mathscr{F}$ is a g-filter then $\phi_{\mathscr{F}}\colon \mathcal{M} \to \mathscr{I}(\mathscr{F})$ defined by

$$\phi_{\mathscr{F}}(x) = \langle \Delta(\mathbf{1}, x) \vee \beta_{\mathscr{F}}(x), x \vee \beta_{\mathscr{F}}(x)\rangle$$

is an isomorphism – the $\mathscr{F}$-presentation of $\mathcal{M}$.

$\mathscr{C}$ gives an isomorphism from $\mathscr{C}(\phi_{\mathscr{F}})\colon \mathscr{C}(\mathcal{M}) \to \mathscr{C}\mathscr{I}(\mathscr{F})$. Note that $\mathscr{C}(\phi_{\mathscr{F}})^{-1} = \mathscr{C}(\phi_{\mathscr{F}}^{-1})$.

Putting this together with $\iota_{\mathscr{F}}$ we have an isomorphism $\iota_{\mathscr{F}}^{-1}\mathscr{C}(\phi_{\mathscr{F}})\colon \mathscr{C}(\mathcal{M}) \to \mathscr{F}$. This induces an isomorphism of automorphism groups

$$\Xi\colon \mathrm{Aut}(\mathscr{C}(\mathcal{M})) \to \mathrm{Aut}(\mathscr{F}) \text{ given by}$$
$$\Xi(\alpha) = \iota_{\mathscr{F}}^{-1}\mathscr{C}(\phi_{\mathscr{F}})\alpha\mathscr{C}(\phi_{\mathscr{F}}^{-1})\iota_{\mathscr{F}}.$$

Lemma 5.2. *Let $x \in \mathscr{F}$. Then*

$$\phi_{\mathscr{F}}^{-1} e_{\mathscr{F}}(x) = x.$$

Proof.

$$\begin{aligned}
\phi_{\mathscr{F}}(x) &= \langle \Delta(\mathbf{1}, x) \vee \beta_{\mathscr{F}}(x), x \vee \beta_{\mathscr{F}}(x)\rangle \\
&= \langle \Delta(\mathbf{1}, x) \vee x, x \vee x\rangle \\
&= \langle \mathbf{1}, x\rangle \\
&= e_{\mathscr{F}}(x).
\end{aligned}$$

□

From this information we are able to identify the image of $\mathscr{C}$.

Theorem 5.3. *Let $\phi \in \mathrm{Aut}(\mathcal{M})$. Let $\phi = \varphi_{\langle \mathscr{F}, \mathscr{G}\rangle} \circ \widehat{\chi}$ where $\chi \in \mathrm{Aut}(\mathscr{F})$. Then*

$$\Xi\mathscr{C}(\phi) = \chi.$$

Proof. Let $x \in \mathscr{F}$. Then

$$\begin{aligned}
\mathscr{C}(\phi_{\mathscr{F}}^{-1})\iota_{\mathscr{F}}(x) &= \mathscr{C}(\phi_{\mathscr{F}}^{-1})\eta_{\mathscr{I}(\mathscr{F})}e_{\mathscr{F}}(x) \\
&= \mathscr{C}(\phi_{\mathscr{F}}^{-1})([e_{\mathscr{F}}(x)]) \\
&= [\phi_{\mathscr{F}}^{-1}e_{\mathscr{F}}(x)] \\
&= [x] && \text{by the lemma.} \\
\mathscr{C}(\phi)([x]) &= \mathscr{C}(\varphi_{\langle \mathscr{F}, \mathscr{G}\rangle})\mathscr{C}(\widehat{\chi})([x]) \\
&= \mathscr{C}(\widehat{\chi})([x]) && \text{as } \varphi_{\langle \mathscr{F}, \mathscr{G}\rangle} \in \ker(\mathscr{C}) \\
&= [\widehat{\chi}(x)] \\
&= [\chi(x)] && \text{as } x \in \mathscr{F} \\
\iota_{\mathscr{F}}^{-1}\mathscr{C}(\phi_{\mathscr{F}})([\chi(x)]) &= \iota_{\mathscr{F}}^{-1}[\phi_{\mathscr{F}}(\chi(x))] \\
&= \iota_{\mathscr{F}}^{-1}[e_{\mathscr{F}}(\chi(x))] \\
&= \iota_{\mathscr{F}}^{-1}\eta_{\mathscr{I}(\mathscr{F})}e_{\mathscr{F}}\chi(x) \\
&= \iota_{\mathscr{F}}^{-1}\iota_{\mathscr{F}}\chi(x) \\
&= \chi(x)
\end{aligned}$$

□

Inner automorphisms of filter algebras are determined by their action on a single g-filter $\mathscr{F}$.

The results cited in section 3, in particular theorem 3.14, shows that any g-filter $\mathscr{G}$ is determined by $\mathscr{F} \cap \mathscr{G}$. This set can be found from the set of fixed points for $\varphi_{\langle \mathscr{F}, \mathscr{G}\rangle}$.

Lemma 5.4. *Let $\mathcal{M} = \mathscr{I}(\mathscr{F})$ be a filter algebra and $\phi = \varphi_{\langle \mathscr{F}, \mathscr{G}\rangle}$ be any inner automorphism. Then*

$$\phi(x) = x \textit{ iff } x \in [\![\mathscr{F} \cap \mathscr{G}]\!].$$

Proof. Let $x \in [\![\mathscr{F} \cap \mathscr{G}]\!]$. Then $\beta_{\mathscr{F}}(x) = \beta_{\mathscr{G}}(x)$ and so $\alpha_{\mathscr{F}}(x) = \alpha_{\mathscr{G}}(x)$. Thus

$$\begin{aligned}
\phi(x) &= \Delta(\beta_{\mathscr{G}}\alpha_{\mathscr{F}}(x), \beta_{\mathscr{G}}(x)) \\
&= \Delta(\alpha_{\mathscr{F}}(x), \beta_{\mathscr{F}}(x)) \\
&= x.
\end{aligned}$$

Conversely, if $\phi(x) = x$ then $x = \phi(x) = \Delta(\beta_{\mathscr{G}}\alpha_{\mathscr{F}}(x), \beta_{\mathscr{G}}(x))$ so that $\beta_{\mathscr{G}}\alpha_{\mathscr{F}}(x) = \alpha_{\mathscr{G}}(x)$. Since $x \leq \alpha_{\mathscr{F}}(x) \wedge \alpha_{\mathscr{G}}(x)$ this implies $\alpha_{\mathscr{F}}(x) = \alpha_{\mathscr{G}}(x)$ is in $\mathscr{F} \cap \mathscr{G}$. But now we have $\beta_{\mathscr{F}}(x) = \Delta(\alpha_{\mathscr{F}}(x), x) = \Delta(\alpha_{\mathscr{G}}(x), x) = \beta_{\mathscr{G}}(x)$ and so $x \in [\![\mathscr{F} \cap \mathscr{G}]\!]$. □

Lemma 5.5. *Let $\mathcal{M} = \mathscr{I}(\mathscr{F})$ be a filter algebra and $\phi = \varphi_{\langle \mathscr{F}, \mathscr{G} \rangle}$ be any inner automorphism. Then*

$$\phi(x) = \Delta(\mathbf{1}, x) \text{ iff } x \in [\![(\mathscr{F} \cap \mathscr{G}) \to \mathscr{F}]\!].$$

Proof. First we recall that $\Delta(\mathbf{1}, \mathscr{G}) \cap \mathscr{F} = (\mathscr{F} \cap \mathscr{G}) \to \mathscr{F}$ – see [2, lemma 5.26].

Let $x \in (\mathscr{F} \cap \mathscr{G}) \to \mathscr{F} = \Delta(\mathbf{1}, \mathscr{G}) \cap \mathscr{F}$. Then $\phi(x) = \beta_{\mathscr{G}}(x)$. As $\Delta(\mathbf{1}, x) \in \mathscr{G}$ we have $\beta_{\mathscr{G}}(x) = \Delta(\mathbf{1}, x)$.

In general we have $x \in [\![(\mathscr{F} \cap \mathscr{G}) \to \mathscr{F}]\!]$ implies $x = \Delta(\alpha_{\mathscr{F}}(x), \beta_{\mathscr{F}}(x))$ where both $\alpha_{\mathscr{F}}(x)$ and $\beta_{\mathscr{F}}(x)$ are in $(\mathscr{F} \cap \mathscr{G}) \to \mathscr{F}$. Then we have

$$\begin{aligned} \phi(x) &= \Delta(\beta_{\mathscr{G}}\alpha_{\mathscr{F}}(x), \beta_{\mathscr{G}}\beta_{\mathscr{F}}(x)) \\ &= \Delta(\Delta(\mathbf{1}, \alpha_{\mathscr{F}}(x)), \Delta(\mathbf{1}, \beta_{\mathscr{F}}(x))) \\ &= \Delta(\mathbf{1}, \Delta(\alpha_{\mathscr{F}}(x), \beta_{\mathscr{F}}(x))) \\ &= \Delta(\mathbf{1}, x). \end{aligned}$$

Conversely, if $\phi(x) = \Delta(\mathbf{1}, x)$ then we have $\Delta(\mathbf{1}, x) = \phi(x) = \Delta(\beta_{\mathscr{G}}\alpha_{\mathscr{F}}(x), \beta_{\mathscr{G}}(x))$ so that $\beta_{\mathscr{G}}\alpha_{\mathscr{F}}(x) = \alpha_{\mathscr{G}}(\Delta(\mathbf{1}, x))$. Since $\beta_{\mathscr{G}}\alpha_{\mathscr{F}}(x) \sim \Delta(\mathbf{1}, \alpha_{\mathscr{F}}(x))$ and $\Delta(\mathbf{1}, x) \leq \alpha_{\mathscr{G}}(\Delta(\mathbf{1}, x)) \wedge \Delta(\mathbf{1}, \alpha_{\mathscr{F}}(x))$ this gives $\alpha_{\mathscr{G}}(\Delta(\mathbf{1}, x)) = \Delta(\mathbf{1}, \alpha_{\mathscr{F}}(x))$. Now we have $x = \Delta(\alpha_{\mathscr{F}}(x), \Delta(\mathbf{1}, \beta_{\mathscr{G}}(x)))$ so that $\beta_{\mathscr{F}}(x) = \Delta(\alpha_{\mathscr{F}}(x), x) = \Delta(\mathbf{1}, \beta_{\mathscr{G}}(x))$. Thus $x \in [\![\Delta(\mathbf{1}, \mathscr{G}) \cap \mathscr{F}]\!]$. □

5.2. **Localizing.** The main technique we will use to derive information about automorphism is localization. Previously we have localized at a point to obtain interval algebras of the form $\mathcal{L}_x$ that are upwards-closed and contain x. Now we need more closure, so we will use a localization that produces filter algebras.

The essential use of localization is contained in the next two results.

Theorem 5.6. *Let $\mathcal{L}$ be a cubic implication algebra and $\mathcal{M}$ be any upwards-closed subalgebra. Then $\mathscr{C}(incl_{\mathcal{M}}) = incl_{\mathscr{C}(\mathcal{M})}$ and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathrm{incl}_{\mathcal{M}}} & \mathcal{L} \\ \Big\downarrow{\scriptstyle \eta_{\mathcal{M}}} & & \Big\downarrow{\scriptstyle \eta_{\mathcal{L}}} \\ \mathscr{C}(\mathcal{M}) & \xrightarrow[\mathrm{incl}_{\mathscr{C}(\mathcal{M})}]{} & \mathscr{C}(\mathcal{L}) \end{array}$$

Proof. The commutativity of the diagram follows from the first result as η is a natural transformation from ID to $\mathscr{C}$.

We show that if $x \in \mathcal{M}$ then $[x]_{\mathcal{M}} = [x]_{\mathcal{L}}$.

Let $y \in [x]_{\mathcal{M}}$. Then $x \sim y$ in $\mathcal{M}$. But this happens iff $\Delta(x \vee y, y) = x$ which is true in $\mathcal{M}$ iff it is true in $\mathcal{L}$. Thus $y \in [x]_{\mathcal{L}}$.

Let $y \in [x]_{\mathcal{L}}$. Then $x \vee y \in \mathcal{M}$ as $\mathcal{M}$ is upwards-closed. But then $\Delta(x \vee y, x) = y$ is also in $\mathcal{M}$ and so $y \in [x]_{\mathcal{M}}$. □

Corollary 5.7. *Let $f\colon \mathcal{L}_1 \to \mathcal{L}_2$ be a cubic homomorphism. Let $\mathcal{M}$ be any upwards-closed subalgebra of $\mathcal{L}_1$. Then*

$$\mathscr{C}(f \restriction \mathcal{M}) = \mathscr{C}(f) \restriction \mathscr{C}(\mathcal{M}).$$

Proof. $f \restriction \mathcal{M} = f \circ \mathrm{incl}_{\mathcal{M}}$ so that

$$\begin{aligned}\mathscr{C}(f \restriction \mathcal{M}) &= \mathscr{C}(f \circ \mathrm{incl}_{\mathcal{M}}) \\ &= \mathscr{C}(f) \circ \mathscr{C}(\mathrm{incl}_{\mathcal{M}}) \\ &= \mathscr{C}(f) \circ \mathrm{incl}_{\mathscr{C}(\mathcal{M})} \\ &= \mathscr{C}(f) \restriction \mathscr{C}(\mathcal{M}).\end{aligned}$$

□

Now here is the localization process we need.

Theorem 5.8. *Let $\mathcal{M}$ be any MR-algebra. Let X be a countable subset of $\mathcal{M}$ and $\mathcal{G}$ be any countable subgroup of* $\mathrm{Aut}(\mathcal{M})$*. Then there is a subalgebra $\mathcal{L}$ of $\mathcal{M}$ such that*

(a) $\mathcal{L}$ is an upwards-closed MR-subalgebra that is countably presented;
(b) $X \subseteq \mathcal{L}$;
(c) if $\phi \in \mathcal{G}$ then $\phi \restriction \mathcal{L}$ is in $\mathrm{Aut}(\mathcal{L})$.

Proof. Define Z inductively by:

$$\begin{aligned}Z_0 &= X \\ Z_{2n+1} &= \text{the caret-closure of } Z_{2n} \\ Z_{2n+2} &= \left\{\phi^i(y) \,\middle|\, y \in Z_{2n+1},\ \phi \in \mathcal{G},\ i \in \omega\right\} \\ Z &= \bigcup_{m\in\omega} Z_m.\end{aligned}$$

Now let

$$\mathcal{L} = \bigcup_{z\in Z} \mathcal{M}_z.$$

Then clearly $\mathcal{L}$ is upwards-closed. It is easy to see that $X \subseteq Z_m \subseteq Z$ for all m so that $X \subseteq \mathcal{L}$. Also Z is countable so that $\mathcal{L}$ is countably presented.

$\mathcal{L}$ is caret-closed – as if $x, y \in \mathcal{L}$ then let $a \preccurlyeq x$ and $b \preccurlyeq y$ for some $a, b \in Z_m$. Then m odd implies $a \mathbin{\hat{}} b \in Z_m$, else $a \mathbin{\hat{}} b \in Z_{m+1}$. Thus $a \mathbin{\hat{}} b \preccurlyeq x \mathbin{\hat{}} y$ and so $x \mathbin{\hat{}} y \in \mathcal{L}$.

Let $\phi \in \mathcal{G}$. Then $\mathcal{L}$ is ϕ-closed – as if $a \in Z_m$ and $a \preccurlyeq z$ then either $\phi(a) \in Z_m$ (if m is even) or $\phi(a) \in Z_{m+1}$. Now we have $\phi(a) \preccurlyeq \phi(z)$ so that $\phi(z) \in \mathcal{L}$.

Let $\phi \in \mathcal{G}$. Then $\phi \restriction \mathcal{L}$ is clearly a one-one homomorphism from $\mathcal{L}$ to $\mathcal{L}$. As $\phi^{-1} \in \mathcal{G}$ we know that $\phi \restriction \mathcal{L}$ is also onto. Thus $\phi \restriction \mathcal{L}$ is in $\mathrm{Aut}(\mathcal{L})$. □

Corollary 5.9. *Let $\mathcal{M}$ be any MR-algebra. Let X be a countable subset of $\mathcal{M}$ and $\mathcal{X}$ be any countable subset of* $\mathrm{Aut}(\mathcal{M})$*. Then there is a subalgebra $\mathcal{L}$ of $\mathcal{M}$ such that*

(a) $\mathcal{L}$ is an upwards-closed MR-subalgebra that is countably presented;
(b) $X \subseteq \mathcal{L}$;
(c) if $\phi \in \mathcal{X}$ then $\phi \restriction \mathcal{L}$ is in $\mathrm{Aut}(\mathcal{L})$.

Proof. Let $\mathcal{G}$ be the subgroup of $\mathrm{Aut}(\mathcal{M})$ generated by $\mathcal{X}$ and apply the theorem. $\square$

5.3. **Inner Automorphisms.** We wish to analyze the kernel of $\mathscr{C}$. Generalizing from filter algebras we define the group of inner automorphisms.

Definition 5.10. *The group* $\ker(\mathscr{C})$ *is called the group of* inner automorphisms *denoted by* $\mathrm{Inn}(\mathcal{M})$.

Earlier (lemma 5.4 and above) we showed that inner automorphisms on filter algebras are determined by their set of fixed points. We will show that this is always true.

Definition 5.11. *Let* $\phi \in \mathrm{Inn}(\mathcal{M})$. *Let*

$$\mathcal{M}_\phi = \{x \mid \phi(x) = x\}.$$

5.4. 2**-torsion.** Let $\mathcal{M}$ be any MR-algebra and let $\alpha \in \mathrm{Aut}(\mathcal{M})$ be an inner automorphism. We wish to show that $\alpha^2 = \mathrm{id}$.

Theorem 5.12. *Let α be any inner automorphism of $\mathcal{M}$. Then $\alpha^2 = id$.*

Proof. Let $x \in \mathcal{M}$. We show that $\alpha^2(x) = x$.

Let $\mathcal{L}$ be as provided by corollary 5.9 for x and α. Then since we have (by corollary 5.7) $\mathscr{C}(\alpha \restriction \mathcal{L}) = \mathscr{C}(\alpha) \restriction \mathscr{C}(\mathcal{L})$ and $\mathscr{C}(\alpha) = \mathrm{id}$ we know that $\alpha \restriction \mathcal{L}$ is an inner automorphism of $\mathcal{L}$. So let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathcal{L}$-g-filters such $\alpha \restriction \mathcal{L} = \varphi_{\langle \mathscr{F}, \mathscr{G} \rangle}$. Then we have

$$\alpha^2(x) = \varphi^2_{\langle \mathscr{F}, \mathscr{G} \rangle}(x) = x.$$

$\square$

Thus $\mathrm{Inn}(\mathcal{M})$ is an abelian group normal in $\mathrm{Aut}(\mathcal{M})$.

6. The Group of Inner Automorphisms

The technique of localization, as used above, establishes more about the group of inner automorphisms. Here we will use it to show how to recover ϕ from $\mathcal{M}_\phi$ and hence that $\mathrm{Inn}(\mathcal{M}) \simeq \mathrm{Inn}(\mathscr{I}\mathscr{C}(\mathcal{M}))$.

Lemma 6.1. *Let $\phi \in \mathrm{Inn}(\mathcal{M})$. Then $\mathcal{M}_\phi$ is an upwards-closed MR-subalgebra of $\mathcal{M}$.*

Proof. Let $x \in \mathcal{M}_\phi$ and let $\mathcal{L}$ be as given by corollary 5.9 for x and ϕ. Then (as in theorem 5.12) we know that $\phi \restriction \mathcal{L}$ is an inner automorphism of $\mathcal{L}$ – say it equals $\varphi_{\langle \mathscr{F}, \mathscr{G} \rangle}$. Then

$$\begin{aligned} \mathcal{M}_\phi \cap \mathcal{L} &= \{x \in \mathcal{L} \mid \phi(x) = x\} \\ &= [\![\mathscr{F} \cap \mathscr{G}]\!] && \text{by lemma 5.4} \end{aligned}$$

which is upwards-closed. Thus $[x, \mathbf{1}] \subseteq \mathcal{M}_\phi$. $\square$

The sets $\mathcal{M}_\phi$ are non-trivial. For example, if $x \in \mathcal{M}$ then $x \vee \phi(x)$ is in $\mathcal{M}_\phi$ – since $\phi^2 = \mathrm{id}$. Also $x \vee \Delta(\mathbf{1}, \phi(x))$ is not in $\mathcal{M}_\phi$ unless $x = \mathbf{1}$.

Theorem 6.2. *Let ϕ_1 and ϕ_2 be in* $\mathrm{Inn}(\mathcal{M})$. *Then*

$$\mathcal{M}_{\phi_1} = \mathcal{M}_{\phi_2} \text{ implies } \phi_1 = \phi_2.$$

Proof. Let $x \in \mathcal{M}$. Let $\mathcal{L}$ be as given by corollary 5.9 for x and $\{\phi_1, \phi_2\}$. Let $\mathscr{F}, \mathscr{G}_1$ and $\mathscr{G}_2$ be $\mathcal{L}$-g-filters such that $\phi_i \restriction \mathcal{L} = \varphi_{\langle \mathscr{F}, \mathscr{G}_i \rangle}$. Then we have $\mathcal{M}_{\phi_i} \cap \mathcal{L} = [\![\mathscr{F} \cap \mathscr{G}_i]\!]$ and as $\mathcal{M}_{\phi_1} = \mathcal{M}_{\phi_2}$ we have $\mathscr{F} \cap \mathscr{G}_1 = \mathscr{F} \cap \mathscr{G}_2$. But then $\mathscr{G}_1 = \Delta(\mathscr{F} \cap \mathscr{G}_1, \mathscr{F}) = \Delta(\mathscr{F} \cap \mathscr{G}_2, \mathscr{F}) = \mathscr{G}_2$ – seetheorem 3.14.

Hence $\phi_1 \restriction \mathcal{L} = \phi_2 \restriction \mathcal{L}$ and so $\phi_1(x) = \phi_2(x)$.

Since we can do this for any $x \in \mathcal{M}$ we have $\phi_1 = \phi_2$. □

This theorem gives only a suggestion of how to recover ϕ from $\mathcal{M}_\phi$. We'll now show how to fully recover ϕ, as a prelude to showing that $\mathrm{Inn}(\mathcal{M}) \simeq \mathrm{Inn}(\mathscr{I}\mathscr{C}(\mathcal{M}))$.

First we note that upwards closed subalgebras are completely determined by their collapses.

Lemma 6.3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two upwards-closed subalgebras of a cubic implication algebra $\mathcal{M}$. Then*

$$\mathcal{L}_1 = \mathcal{L}_2 \textit{ iff } \mathscr{C}(\mathcal{L}_1) = \mathscr{C}(\mathcal{L}_2).$$

Proof. This is because $\mathcal{L}_i = \bigcup \{[x] \mid [x] \in \mathscr{C}(\mathcal{L}_i)\}$. □

The main fact we know about ϕ is that $x \sim \phi(x)$ for all x. This is another way of viewing $\mathscr{C}(\phi) = \mathrm{id}$. It implies that

$$x = (x \vee \phi(x)) \wedge (x \vee \Delta(\mathbf{1}, \phi(x))) \tag{2}$$

$$\phi(x) = (x \vee \phi(x)) \wedge (\Delta(\mathbf{1}, x) \vee \phi(x)) \tag{3}$$

$$\Delta(\mathbf{1}, x) \vee \phi(x) = \Delta(\mathbf{1}, x \vee \Delta(\mathbf{1}, \phi(x))) \tag{4}$$

for any x. Since we already know that $x \vee \phi(x)$ is in $\mathcal{M}_\phi$ we need to look at the other half of the first equation. The other two equations suggest the recovery of ϕ – it's the identity on $\mathcal{M}_\phi$ and $\Delta(\mathbf{1}, \bullet)$ on the other side. So we need to identify the other side!

Definition 6.4. *Let $\phi \in \mathrm{Inn}(\mathcal{M})$. Let*

$$D_\phi = \eta_{\mathcal{M}}^{-1}[\mathscr{C}(\mathcal{M}_\phi) \to \mathscr{C}(\mathcal{M})].$$

Lemma 6.5. *Let $x \in \mathcal{M}$ be arbitrary. Then $\Delta(\mathbf{1}, x) \vee \phi(x) \in D_\phi$.*

Proof. Let $z = \eta_{\mathcal{M}}(\Delta(\mathbf{1}, x) \vee \phi(x))$. We want to show that $z \in \mathscr{C}(\mathcal{M}_\phi) \to \mathscr{C}(\mathcal{M})$ – that is for any $[y] \in \mathscr{C}(\mathcal{M}_\phi)$ we have $z \vee [y] = \mathbf{1}$. This is equivalent to showing that $\eta_{\mathcal{M}}(y * (\Delta(\mathbf{1}, x) \vee \phi(x))) = \mathbf{1}$ and as $\eta_{\mathcal{M}}^{-1}[\mathbf{1}] = \{\mathbf{1}\}$ we need to show that $y * (\Delta(\mathbf{1}, x) \vee \phi(x)) = \mathbf{1}$.

$$\begin{aligned} y * (\Delta(\mathbf{1}, x) \vee \phi(x)) &= \Delta(\Delta(\mathbf{1}, x) \vee \phi(x) \vee y, y) \vee (\Delta(\mathbf{1}, x) \vee \phi(x)) \\ \Delta(\mathbf{1}, x) \vee \phi(x) \vee y &\geq y \end{aligned}$$

so it must be in $\mathcal{M}_\phi$ and therefore

$$\begin{aligned} \Delta(\mathbf{1}, x) \vee \phi(x) \vee y &= \phi(\Delta(\mathbf{1}, x) \vee \phi(x) \vee y) \\ &= \Delta(\mathbf{1}, \phi(x)) \vee x \vee y \\ \text{and so } \Delta(\mathbf{1}, x) \vee \phi(x) \vee y &= (\Delta(\mathbf{1}, x) \vee \phi(x) \vee y) \vee (\Delta(\mathbf{1}, \phi(x)) \vee x \vee y) \\ &= \mathbf{1}. \end{aligned}$$

Thus

$$y * (\Delta(\mathbf{1}, x) \vee \phi(x)) = \Delta(\mathbf{1}, y) \vee \Delta(\mathbf{1}, x) \vee \phi(x).$$

As this is also in $\mathcal{M}_\phi$ it must also equal $\mathbf{1}$. □

Lemma 6.6. *Let $z \in D_\phi$. Then $\phi(z) = \Delta(\mathbf{1}, z)$.*

Proof. Let $\mathcal{L}$ be as given by corollary 5.9 for z and ϕ. Let $\phi \restriction \mathcal{L} = \varphi_{\langle \mathscr{F}, \mathscr{G} \rangle}$. We know that $\mathcal{M}_\phi \cap \mathcal{L} = [\![\mathscr{F} \cap \mathscr{G}]\!]$ so that – in $\mathcal{L}$ – we have $D_{\phi \restriction \mathcal{L}} = [\![(\mathscr{F} \cap \mathscr{G}) \to \mathscr{F}]\!]$ and that $\phi \restriction \mathcal{L}$ is equal to $\Delta(\mathbf{1}, \bullet)$ on this set – by lemma 5.5.

Since $z \in D_\phi$ we have $\eta_{\mathcal{M}}(z) = \eta_{\mathcal{L}}(z) \in \mathscr{C}(\mathcal{M}_\phi \cap \mathcal{L}) \to \mathscr{C}(\mathcal{L})$ and so $z \in D_{\phi \restriction \mathcal{L}}$. Thus $\phi(z) = (\phi \restriction \mathcal{L})(z) = \Delta(\mathbf{1}, z)$. □

Corollary 6.7. $\mathcal{M}_\phi \cap D_\phi = \{\mathbf{1}\}$.

Proof. Let $x \in \mathcal{M}_\phi \cap D_\phi$. Then we have $x = \phi(x) = \Delta(\mathbf{1}, x)$ so that $x = x \vee \Delta(\mathbf{1}, x) = \mathbf{1}$. □

Corollary 6.8. *If $x \in \mathcal{M}_\phi$ and $y \in D_\phi$ then $x \wedge \Delta(\mathbf{1}, y)$ exists.*

Proof. This follows from the MR-axiom as $x \vee y \in \mathcal{M}_\phi \cap D_\phi$ and so equals $\mathbf{1}$. □

Lemma 6.9. *Let $z \in \mathcal{M}$ be arbitrary. Then there is a unique pair $\langle z_0, z_1 \rangle \in \mathcal{M}_\phi \times D_\phi$ such that*

$$z = z_0 \wedge z_1.$$

Proof. We know that $z \sim \phi(z)$ so that $z = (z \vee \phi(z)) \wedge (z \vee \Delta(\mathbf{1}, \phi(z)))$. From above we have $z \vee \phi(z) \in \mathcal{M}_\phi$ and $\Delta(\mathbf{1}, z) \vee \phi(z) \in D_\phi$. As D_ϕ is $\Delta(\mathbf{1}, \bullet)$-closed we have $z \vee \Delta(\mathbf{1}, \phi(z)) \in D_\phi$.

Suppose that $z = z_0 \wedge z_1$ with $\langle z_0, z_1 \rangle \in \mathcal{M}_\phi \times D_\phi$. Then we have

$$\begin{aligned} z \vee \phi(z) &= (z_0 \wedge z_1) \vee (\phi(z_0) \wedge \phi(z_1)) \\ &= (z_0 \wedge z_1) \vee (z_0 \wedge \Delta(\mathbf{1}, z_1)) \\ &= z_0 \wedge (z_1 \vee \Delta(\mathbf{1}, z_1)) \\ &= z_0. \end{aligned}$$

Likewise we have $z \vee \Delta(\mathbf{1}, \phi(z)) = z_1$. □

Lemma 6.10. *Let $z \in \mathcal{M}$ be arbitrary and $\langle z_0, z_1 \rangle \in \mathcal{M}_\phi \times D_\phi$ be such that*

$$z = z_0 \wedge z_1.$$

Then

$$\phi(z) = z_0 \wedge \Delta(\mathbf{1}, z_1).$$

Proof. Since ϕ preserves meets that exist and from the definition of $\mathcal{M}_\phi$ and lemma 6.6. □

This completes our recovery from ϕ from $\mathcal{M}_\phi$. But we need more in order to prove that

$$\mathrm{Inn}(\mathcal{M}) \simeq \mathrm{Inn}(\mathscr{I}\mathscr{C}(\mathcal{M})).$$

Lemma 6.11. *$\mathscr{C}(\mathcal{M}_\phi)$ is $\mathscr{C}(\mathcal{M})$-Boolean.*

Proof. Let $z \in \mathcal{M}$. Then

$$\begin{aligned} z &= (z \vee \phi(z)) \wedge (z \vee \Delta(\mathbf{1}, \phi(z))) \\ &= (z \vee \phi(z)) \;\hat{}\; (\Delta(\mathbf{1}, z) \vee \phi(z)). \end{aligned}$$

Thus we have $[z] = [z \vee \phi(z)] \wedge [\Delta(\mathbf{1}, z) \vee \phi(z)]$.

We know that $z \vee \phi(z) \in \mathcal{M}_\phi$ and so $[z \vee \phi(z)] \in \mathscr{C}(\mathcal{M}_\phi)$. Also $\Delta(\mathbf{1}, z) \vee \phi(z) \in D_\phi$ so that $[\Delta(\mathbf{1}, z) \vee \phi(z)] \in \mathscr{C}(\mathcal{M}_\phi) \to \mathscr{C}(\mathcal{M})$. □

Lemma 6.12. *Let $\mathscr{F}$ be a g-filter in some filter algebra $\mathcal{L}$, and $\mathscr{G}$, $\mathscr{H}$ be two subfilters of $\mathscr{F}$. If $\mathscr{G}$ is $\mathscr{F}$-Boolean then $\mathscr{G} \cap \mathscr{H}$ is $\mathscr{H}$-Boolean.*

Proof. Firstly we see that if $s \in (\mathscr{G} \to \mathscr{F}) \cap \mathscr{H}$ then for all $g \in \mathscr{G} \cap \mathscr{H}$ we have $s \vee g = \mathbf{1}$. Thus

$$(\mathscr{G} \to \mathscr{F}) \cap \mathscr{H} \subseteq (\mathscr{G} \cap \mathscr{H}) \to \mathscr{H}.$$

Now if $h \in \mathscr{H}$ then there is some $g \in \mathscr{G}$ and $k \in \mathscr{G} \to \mathscr{F}$ such that $h = g \wedge k$. As $h \leq g$ we have $g \in \mathscr{G} \cap \mathscr{H}$. And $h \leq k$ implies $k \in (\mathscr{G} \to \mathscr{F}) \cap \mathscr{H}$ and so $k \in (\mathscr{G} \cap \mathscr{H}) \to \mathscr{H}$. □

Lemma 6.13. *Let $\mathscr{F}$ be a g-filter in some filter algebra $\mathcal{L}$, and $\mathscr{G}$ be a $\mathscr{F}$-Boolean subfilter of $\mathscr{F}$. Let $f \in \mathscr{F}$. Then $\mathscr{G} \cap [f, \mathbf{1}]$ is principal.*

Proof. Let $f = g \wedge h$ for $g \in \mathscr{G}$ and $h \in \mathscr{G} \to \mathscr{F}$. Then we have $[g, \mathbf{1}] \subseteq \mathscr{G} \cap [f, \mathbf{1}]$.

If $x \in \mathscr{G} \cap [f, \mathbf{1}]$ then

$$\begin{aligned} x &= x \vee f \\ &= x \vee (g \wedge h) \\ &= (x \vee g) \wedge (x \vee h) \\ &= x \vee g && \text{as } x \vee h = \mathbf{1}. \end{aligned}$$

Thus $x \geq g$, and so $[g, \mathbf{1}] = \mathscr{G} \cap [f, \mathbf{1}]$. □

Lemma 6.14. *Let $\mathcal{L} = \mathcal{L}_a$ be an interval algebra. Let $g \geq a$ and $h = g \to a$. Then for any $z \in \mathcal{L}$*

$$z = (z \vee \Delta(g \vee z, g)) \wedge (z \vee \Delta(h \vee z, h)).$$

Proof. We work in an interval algebra $\mathscr{I}(B)$. Wolog $a = [0, 0]$ so that $g = [0, g]$ and $h = [0, \overline{g}]$. Let $z = [z_0, z_1]$. Then we have

$$\begin{aligned} g \vee z &= [0, g \vee z_1] \\ h \vee z &= [0, \overline{g} \vee z_1] \\ \Delta(g \vee z, g) &= [0 \vee ((g \vee z_1) \wedge \overline{g}), 0 \vee ((g \vee z_1) \wedge \overline{0})] \\ &= [\overline{g} \wedge z_1, g \vee z_1] \\ \Delta(h \vee z, h) &= [g \wedge z_1, \overline{g} \vee z_1] \\ z \vee \Delta(g \vee z, g) &= [\overline{g} \wedge z_0, g \vee z_1] \\ z \vee \Delta(h \vee z, h) &= [g \wedge z_0, \overline{g} \vee z_1] \\ (z \vee \Delta(g \vee z, g)) \wedge (z \vee \Delta(h \vee z, h)) &= [(\overline{g} \wedge z_0) \vee (g \wedge z_0), (g \vee z_1) \wedge (\overline{g} \vee z_1)] \\ &= [z_0, z_1]. \end{aligned}$$

□

Theorem 6.15. *Let $\mathscr{G}$ be a $\mathscr{C}(\mathcal{M})$-Boolean filter. Let*

$$\begin{aligned} S_1 &= \eta_{\mathcal{M}}^{-1}[\mathscr{G}],\\ S_2 &= \eta_{\mathcal{M}}^{-1}[\mathscr{G} \to \mathscr{C}(\mathcal{M})]. \end{aligned}$$

Then

1. $S_1 \cap S_2 = \{\mathbf{1}\}$.
2. *For all $x \in \mathcal{M}$ there are unique $x_1 \in S_1$ and $x_2 \in S_2$ with $x = x_1 \wedge x_2$.*
3. *If we define $\phi_{\mathscr{G}} \colon \mathcal{M} \to \mathcal{M}$ by*

$$\phi_{\mathscr{G}}(x) = x_1 \wedge \Delta(\mathbf{1}, x_2) \tag{5}$$

then
   (a) *$\phi_{\mathscr{G}}$ is a well-defined cubic automorphism of $\mathcal{M}$;*
   (b) *$\phi_{\mathscr{G}}$ is an inner automorphism of $\mathcal{M}$;*
   (c) *$\mathcal{M}_{\phi_{\mathscr{G}}} = S_1$.*

Proof. B1. Let $x \in S_1 \cap S_2$. Then we have $[x] = [x] \vee [x] = \mathbf{1}$ so that $x \in \eta_{\mathcal{M}}^{-1}[\mathbf{1}] = \{\mathbf{1}\}$.

Note that this implies $x \wedge \Delta(\mathbf{1}, y)$ exists for all $x \in S_1$ and $y \in S_2$ from the MR-axiom and $x \vee y = \mathbf{1}$.

B2. Let $x \in \mathcal{M}$. Then $[x] = \mathbf{x}_1 \wedge \mathbf{x}_2$ for some $\mathbf{x}_1 \in \mathscr{G}$ and $\mathbf{x}_2 \in \mathscr{G} \to \mathscr{C}(\mathcal{M})$ – as $\mathscr{G}$ is a $\mathscr{C}(\mathcal{M})$-Boolean filter.

Let $x_1' \in S_1$ and $x_2' \in S_2$ be such that $\mathbf{x}_i = [x_i']$. Then we have $x \sim x_1' \hat{\ } x_2'$. Hence

$$\begin{aligned} x &= \Delta(x \vee (x_1' \hat{\ } x_2'), x_1' \hat{\ } x_2')\\ &= \Delta(x \vee (x_1' \hat{\ } x_2'), x_1' \wedge \Delta(x_1' \vee x_2', x_2'))\\ &= \Delta(x \vee (x_1' \hat{\ } x_2'), x_1') \wedge \Delta(x \vee (x_1' \hat{\ } x_2'), \Delta(x_1' \vee x_2', x_2')). \end{aligned}$$

As $\Delta(x \vee (x_1' \hat{\ } x_2'), x_1') \sim x_1'$ we have $\Delta(x \vee (x_1' \hat{\ } x_2'), x_1') \in S_1$. Likewise $\Delta(x \vee (x_1' \hat{\ } x_2'), \Delta(x_1' \vee x_2', x_2'))$ is in S_2.

Suppose that $x = x_1 \wedge x_2 = y_1 \wedge y_2$ with x_1, y_1 in S_1 and x_2, y_2 in S_2. Then we have

$$\begin{aligned} x_1 &= x_1 \vee x\\ &= x_1 \vee (y_1 \wedge y_2)\\ &= (x_1 \vee y_1) \wedge (x_1 \vee y_2)\\ &= x_1 \vee y_1 && \text{as } x_1 \vee y_2 \in S_1 \cap S_2. \end{aligned}$$

Thus $x_1 \geq y_1$. Dually $x_1 \leq y_1$.

In a similar way we have $x_2 = y_2$.

B3. (a) This part we do by localizing. Let $x, y \in \mathcal{M}$ and let $\mathcal{L} = \mathcal{M}_{x \hat{\ } y}$. Then we have $\mathscr{C}(\mathcal{L}) = [[x] \wedge [y], \mathbf{1}]$ so we let $\mathscr{G}_{\mathcal{L}} = \mathscr{G} \cap [[x] \wedge [y], \mathbf{1}] = [[g], \mathbf{1}]$ for some $g \geq x \hat{\ } y$. Also we note that $S_{1,\mathcal{L}} = \eta_{\mathcal{M}}^{-1}[\mathscr{G}_{\mathcal{L}}] = S_1 \cap \mathcal{L} = \mathcal{L}_g$, and $S_2 \cap \mathcal{L} = \mathcal{L}_{g \to (x \hat{\ } y)}$. Let $h = g \to (x \hat{\ } y)$ and $a = x \hat{\ } y$.

Thus we have for any $z \in \mathcal{L}$ that

$$z = (z \vee \Delta(z \vee g, g)) \wedge (z \vee \Delta(z \vee h, h))$$

is the decomposition given by part (2) above. Let $\phi_g = \phi_{\mathscr{G}} \restriction \mathcal{L}$ so that

$$\phi_{\mathscr{G}}(w) = (w \vee \Delta(x \vee g, g)) \wedge \Delta(\mathbf{1}, w \vee \Delta(x \vee h, h))$$

for any $w \in \mathcal{L}$.
We know that in any cubic implication algebra, if $a \sim b$ then $f_{ab}\colon [a, \mathbf{1}] \to [b, \mathbf{1}]$ defined by

$$f_{ab}(w) = (w \vee b) \wedge (\Delta(\mathbf{1}, w) \vee b)$$

extends to an inner automorphism of $\mathcal{L}_a$ by

$$\widehat{f}_{ab}(z) = f_{ab}(z \vee a) \wedge \Delta\Big(\mathbf{1}, f_{ab}\big((\Delta(\mathbf{1}, z) \vee a) \to a\big) \to b\Big).$$

Let

$$b = \Delta(g, a).$$

Then we have
$$\begin{aligned} b &= \Delta(g, a) \\ &= g \wedge \Delta(\mathbf{1}, g \to a) \\ &= (g \vee a) \wedge \Delta(\mathbf{1}, h \vee a) = \phi_{\mathscr{G}}(a) \end{aligned}$$
is the image of the mapping restricted to $\mathcal{L}$.
We claim that ϕ_g is exactly the inner automorphism $\widehat{f}_{ab}$. By lemma 1.11 it suffices to show that

$$\widehat{f}_{ab}(z) \vee \Delta(g, a) = \phi_g(z) \vee \Delta(g, a) \tag{6}$$
$$\widehat{f}_{ab}(z) \vee \Delta(\mathbf{1}, \Delta(g, a)) = \phi_g(z) \vee \Delta(\mathbf{1}, \Delta(g, a)) \tag{7}$$

for all z. Since we are working in an interval algebra, we will use intervals from a Boolean algebra to do this.
Let $z = [z_0, z_1]$. Wolog $a = [0, 0]$, $g = [0, g]$ and $h = [0, \overline{g}]$. Then $b = \Delta(g, a) = [g, g]$. As above (lemma 6.14) we have

$$\begin{aligned} z \vee \Delta(g \vee z, g) &= [\overline{g} \wedge z_0, g \vee z_1] \\ z \vee \Delta(h \vee z, h) &= [g \wedge z_0, \overline{g} \vee z_1] \end{aligned}$$

and so

$$\begin{aligned} \phi_g(z) &= [\overline{g} \wedge z_0, g \vee z_1] \wedge [g \wedge \overline{z}_1, \overline{g} \vee \overline{z}_0] \\ &= [(\overline{g} \wedge z_0) \vee (g \wedge \overline{z}_1), (g \vee z_1) \wedge (\overline{g} \vee \overline{z}_0)] \\ \phi_g(z) \vee \Delta(g, a) &= \phi_g(z) \vee [g, g] = [g \wedge \overline{z}_1, g \vee z_1] \\ \phi_g(z) \vee \Delta(\mathbf{1}, \Delta(g, a)) &= \phi_g(z) \vee [\overline{g}, \overline{g}] = [\overline{g} \wedge z_0, \overline{g} \vee \overline{z}_0]. \end{aligned}$$

By general theory we have

$$\begin{aligned} \widehat{f}_{ab}(z) \vee \Delta(g, a) &= f_{ab}(z \vee a) \\ \widehat{f}_{ab}(z) \vee \Delta(\mathbf{1}, \Delta(g, a)) &= \Delta\Big(\mathbf{1}, f_{ab}\big((\Delta(\mathbf{1}, z) \vee a) \to a\big) \to b\Big). \end{aligned}$$

This gives us

$$\begin{aligned}
\widehat{f}_{ab}(z)\vee\Delta(g,a) &= f_{ab}(z\vee a)\\
&= (z\vee a\vee b)\wedge(\Delta(\mathbf{1},z)\vee\Delta(\mathbf{1},a)\vee b)\\
&= ([z_0,z_1]\vee[0,g])\wedge([\overline{z}_1,\overline{z}_0]\vee[1,1]\vee[g,g])\\
&= [0,z_1\vee g]\wedge[\overline{z}_1\wedge g,\mathbf{1}]\\
&= [\overline{z}_1\wedge g,z_1\vee g]\\
&= \phi_g(z)\vee\Delta(g,a).\\
f_{ab}((\Delta(\mathbf{1},z)\vee a)\to a) &= f_{ab}([0,\overline{z}_0]\to[0,0])\\
&= f_{ab}([0,z_0])\\
&= ([0,z_0]\vee[g,g])\wedge([\overline{z}_0,1]\vee[g,g])\\
&= [0,z_0\vee g]\wedge[\overline{z}_0\wedge g,1]\\
&= [\overline{z}_0\wedge g,z_0\vee g]\\
\widehat{f}_{ab}(z)\vee\Delta(\mathbf{1},\Delta(g,a)) &= \Delta(\mathbf{1},f((\Delta(\mathbf{1},z)\vee a)\to a)\to b)\\
&= \Delta(\mathbf{1},[\overline{z}_0\wedge g,z_0\vee g]\to[g,g])\\
&= \Delta(\mathbf{1},[z_0\wedge g,\overline{z}_0\vee g])\\
&= [z_0\wedge\overline{g},\overline{z}_0\vee\overline{g}]\\
&= \phi_g(z)\vee\Delta(\mathbf{1},\Delta(g,a)).
\end{aligned}$$

It follows from this that $\phi_{\mathscr{G}}$ is a well-defined, one-one, and onto cubic homomorphism.

Well-defined: Immediate from the definition and part (2).

One-one: As if $\phi_{\mathscr{G}}(x)=\phi_{\mathscr{G}}(y)$ we can work as above and see that $\widehat{f}_{ab}(x)=\widehat{f}_{ab}(y)$ so that $x=y$.

Homomorphism: Let x,y be given, and use $\mathcal{L}$ be constructed with x,y. Then $x\vee y$ and $\Delta(x,y)$ are in $\mathcal{L}$ so we have $\phi_{\mathscr{G}}(x\vee y)=\widehat{f}_{ab}(x\vee y)=\widehat{f}_{ab}(x)\vee\widehat{f}_{ab}(y)=\phi_{\mathscr{G}}(x)\vee\phi_{\mathscr{G}}(y)$. Likewise Δ is preserved.

Onto: To show this we note that $\phi_{\mathscr{G}}^2=\mathrm{id}$ as if $x=x_1\wedge x_2$ with $x_i\in S_i$ then the unique representation of $\phi_{\mathscr{G}}(x)$ is $x_1\wedge\Delta(\mathbf{1},x_2)$. Thus

$$\begin{aligned}
\phi_{\mathscr{G}}^2(x) &= \phi_{\mathscr{G}}(x_1\wedge\Delta(\mathbf{1},x_2))\\
&= x_1\wedge\Delta(\mathbf{1},\Delta(\mathbf{1},x_2))\\
&= x_1\wedge x_2 = x.
\end{aligned}$$

Since this is true, if $x\in\mathcal{M}$ then $x=\phi_{\mathscr{G}}^2(x)$ is in the range of $\phi_{\mathscr{G}}$.

(b) We can argue as above to see that for any $x\in\mathcal{M}$ we have $\phi_{\mathscr{G}}(x)=\widehat{f}_{ab}(x)\sim x$ so that $\mathscr{C}(\phi_{\mathscr{G}})=\mathrm{id}$.

(c) If $x\in S_1$ then $x=x\wedge\mathbf{1}$ so that $\phi_{\mathscr{G}}(x)=x\wedge\Delta(\mathbf{1},\mathbf{1})=x$.
If $\phi_{\mathscr{G}}(x)=x=x_1\wedge x_2$ with $x_i\in S_i$ then we have $x_1\wedge x_2=x_1\wedge\Delta(\mathbf{1},x_2)$. By the uniqueness of representation we have $x_2=\Delta(\mathbf{1},x_2)$ and so $x_2=\mathbf{1}$. Thus $x=x_1\in S_1$.

□

Theorem 6.16.

$$\mathrm{Inn}(\mathcal{M})\simeq\mathrm{Inn}(\mathscr{I}\mathscr{C}(\mathcal{M})).$$

Proof. We know from [2] that $\mathrm{Inn}(\mathscr{I}\mathscr{C}(\mathcal{M}))$ is isomorphic to the group of $\mathscr{C}(\mathcal{M})$-Boolean filters.

We define a mapping Ω from $\mathrm{Inn}(\mathcal{M})$ to this group by

$$\Omega(\phi) = \mathscr{C}(\mathcal{M}_\phi).$$

By lemma 6.11 this filter is $\mathscr{C}(\mathcal{M})$-Boolean. By lemma 6.3 and theorem 6.2 this mapping is one-one. By the last theorem the mapping is onto. We just need to show that it is a homomorphism, i.e. that

$$\mathscr{C}(\mathcal{M}_{\phi_1\phi_2}) = \mathscr{C}(\mathcal{M}_{\phi_1}) + \mathscr{C}(\mathcal{M}_{\phi_2}).$$

First notice that $\mathcal{M}_{\phi_1\phi_2} = \{x \mid \phi_1(x) = \phi_2(x)\}$ – since both maps are their own inverses. By definition

$$\begin{aligned}\mathscr{C}(\mathcal{M}_{\phi_1}) + \mathscr{C}(\mathcal{M}_{\phi_2}) = \Big[\big(\mathscr{C}(\mathcal{M}_{\phi_1}) \to \mathscr{C}(\mathcal{M})\big) \cap \big(\mathscr{C}(\mathcal{M}_{\phi_2}) \to \mathscr{C}(\mathcal{M})\big)\Big] \vee \\ \Big[\mathscr{C}(\mathcal{M}_{\phi_1}) \cap \mathscr{C}(\mathcal{M}_{\phi_2})\Big].\end{aligned}$$

Let $x \in \mathcal{M}$ be arbitrary. Then $x = x_1 \wedge x_2$ for some $x_1 \in \mathcal{M}_{\phi_1}$ and $x_2 \in D_{\phi_1}$. Now let $x_1 = x_{11} \wedge x_{12}$ and $x_2 = x_{21} \wedge x_{22}$ where $x_{i1} \in \mathcal{M}_{\phi_2}$ and $x_{i2} \in D_{\phi_2}$. Then we have

$$\begin{aligned}\phi_1(x) &= x_{11} \wedge x_{12} \wedge \Delta(\mathbf{1}, x_{21}) \wedge \Delta(\mathbf{1}, x_{22})\\ \phi_2(x) &= x_{11} \wedge \Delta(\mathbf{1}, x_{12}) \wedge x_{21} \wedge \Delta(\mathbf{1}, x_{22}).\end{aligned}$$

Then if $\phi_1(x) = \phi_2(x)$ the uniqueness of the representations implies that $x_{12} = \Delta(\mathbf{1}, x_{12})$ and $x_{21} = \Delta(\mathbf{1}, x_{21})$. Thus $x_{12} = x_{21} = \mathbf{1}$. Therefore $x = x_{11} \wedge x_{22}$ with $x_{11} \in \mathcal{M}_{\phi_1} \cap \mathcal{M}_{\phi_2}$ and $x_{22} \in D_{\phi_1} \cap D_{\phi_2}$. As this entails $x = x_{11} \,\hat{}\, \Delta(\mathbf{1}, x_{22})$ we have $[x] = [x_{11}] \wedge [x_{22}]$ and $[x_{11}] \in \Big[\mathscr{C}(\mathcal{M}_{\phi_1}) \cap \mathscr{C}(\mathcal{M}_{\phi_2})\Big]$, $[x_{22}] \in \Big[\big(\mathscr{C}(\mathcal{M}_{\phi_1}) \to \mathscr{C}(\mathcal{M})\big) \cap \big(\mathscr{C}(\mathcal{M}_{\phi_2}) \to \mathscr{C}(\mathcal{M})\big)\Big]$. Thus $[x] \in \mathscr{C}(\mathcal{M}_{\phi_1}) + \mathscr{C}(\mathcal{M}_{\phi_2})$.

Conversely, if $[x] \in \mathscr{C}(\mathcal{M}_{\phi_1}) + \mathscr{C}(\mathcal{M}_{\phi_2})$ implies $[x] = [x_1] \wedge [x_2]$ for some $[x_1]$ in $\Big[\mathscr{C}(\mathcal{M}_{\phi_1}) \cap \mathscr{C}(\mathcal{M}_{\phi_2})\Big]$, $[x_2]$ in $\Big[\big(\mathscr{C}(\mathcal{M}_{\phi_1}) \to \mathscr{C}(\mathcal{M})\big) \cap \big(\mathscr{C}(\mathcal{M}_{\phi_2}) \to \mathscr{C}(\mathcal{M})\big)\Big]$, and (as in previous analyses) we may assume that $x = x_1 \wedge x_2$ where $x_1 \in \mathcal{M}_{\phi_1} \cap \mathcal{M}_{\phi_2}$ and $x_2 \in D_{\phi_1} \cap D_{\phi_2}$. Then we have

$$\phi_1(x) = x_1 \wedge \Delta(\mathbf{1}, x_2)$$

and

$$\phi_2(x) = x_1 \wedge \Delta(\mathbf{1}, x_2).$$

Thus $[x] \in \mathscr{C}(\mathcal{M}_{\phi_1\phi_2})$. □

The isomorphism we have constructed in this theorem comes about in a rather indirect fashion. This is in sharp contrast to earlier isomorphism results that came about via extension of homomorphism results. In this case such extensions seem impossible to obtain.

References

[1] C. G. Bailey and J. S. Oliveira, *An Axiomatization for Cubic Algebras*, Mathematical Essays in Honor of Gian-Carlo Rota (B. E. Sagan and R. P. Stanley, eds.), Birkhaüser, 1998., pp. 305–334.

[2] ______, *Cube-like structures generated by filters*, Algebra Universalis **49** (2003), 129–158.

[3] ______, *Signed Set Families as Posets with Involution*, in preparation.

[4] N. Metropolis and G.-C. Rota, *Combinatorial Structure of the faces of the n-Cube*, SIAM J.Appl.Math. **35** (1978), 689–694.

School of Mathematics, Statistics & Operations Research, Victoria University of Wellington, Wellington, New Zealand,

E-mail address: `Colin.Bailey@vuw.ac.nz`

Pacific Northwest National Laboratories, Richland, Washington, U.S.A.

E-mail address: `Joseph.Oliveira@pnl.gov`